$$d\omega=\frac{\cos\phi\, d\phi}{\sqrt{2}\sqrt{[1-\frac{1}{2}\sin^2\phi)]}}.$$

Whence $A_\theta=\frac{1}{2}\sqrt{2}\int_0^\phi\frac{d\omega}{\sqrt{[1-\frac{1}{2}\sin^2\phi]}}=\frac{1}{2}\sqrt{[2]}F(\frac{1}{2}\sqrt{2},\ \phi)$.

The amplitude ϕ is determined from $\theta=\tan\omega$, $\cos^2\omega=\frac{1}{1+\theta^2}$, $\sin^2\omega=\frac{\theta^2}{1+\theta^2}$, $\sin^2\phi=\frac{2\theta^2}{1+\theta^2}$, $\cos^2\phi=\frac{1-\theta^2}{1+\theta^2}$. Hence, $\tan^2\phi=\frac{2\theta^2}{1-\theta^2}$.

Referring to problem 303,

$$A=\int_0^1\frac{dx}{\sqrt{[1-x^4]}}=\frac{\sqrt{\pi}}{4}\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})},$$

the well known formula $\Gamma(\theta)\Gamma(1-\theta)=\pi/\sin\pi\theta$, gives in our case $\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})=\pi\sqrt{2}$. Whence, $\Gamma(\frac{3}{4})=\frac{\pi\sqrt{2}}{\Gamma(\frac{1}{4})}$, and therefore, $A=\frac{[\Gamma(\frac{1}{4})]^2}{4\sqrt{[2\pi]}}$.

This combined with the result in above, $A=\frac{1}{2}\sqrt{[2]}F^I(\frac{1}{2}\sqrt{2})$ yields $\Gamma(\frac{1}{4})=2\sqrt[4]{[\pi]}\{F^I(\frac{1}{2}\sqrt{2})\}^{\frac{1}{2}}=3.62561$, *0.5593811.

And similarly, $\Gamma(\frac{3}{4})=\frac{1}{2}\sqrt{[2]}\sqrt[4]{[\pi^3]}\{F^I(\frac{1}{2}\sqrt{2})\}^{-\frac{1}{2}}=1.225416$, *0.0882838.

These expressions for $\Gamma(\frac{1}{4})$ and $\Gamma(\frac{3}{4})$ in Legendre's F-functions are to my mind by far the most important consequences of evaluating integral A in gamma-functions. Without this evaluation $\Gamma(\frac{1}{4})$ and $\Gamma(\frac{3}{4})$ can be determined only by computing their natural logarithms by inconvenient series.

Also solved by V. M. Spunar, C. N. Schmall, and J. Scheffer.

MECHANICS.

253. Proposed by W. J. GREENSTREET, M. A., Editor, The Mathematical Gazette, Stroud, England.

R_1 and R_2 are ranges on a horizontal plane of particles projected with given velocity from A on the plane to pass through B. Show that $a(R_1+R_2)-R_1R_2=\frac{a^4}{c^2}$, where $c=AB$ and a is the horizontal projection of AB.

III. Solution by the PROPOSER.

If α, α_1 be angles of projection and β the angle AB makes with the horizontal, and v the velocity of projection, then $\alpha_1=\frac{1}{2}\pi-(\alpha-\beta)$.

And $\cos\beta=a/c$; $R_1.g=a^2\sin 2\alpha$; $R_2.g=a^2\sin 2(\alpha-\beta)$.

$$AB=c=\frac{2v^2}{g}\frac{\cos\alpha\sin(\alpha-\beta)}{\cos\beta}=\frac{2v^2c^2}{g\,a^2}\sin(\alpha-\beta)\cos\alpha.$$

$$\therefore a(R_1+R_2)=\frac{av^2}{g}[\sin 2\alpha+\sin 2(\alpha-\beta)]=\frac{2av^2}{g}\sin 2(\alpha-\beta)\cos\beta$$

$$=\frac{2av^2}{g}[\sin\alpha\cos(\alpha-\beta)+\cos\alpha\sin(\alpha-\beta)]\cos\beta$$

$$=\frac{a^4}{c^2}\left(\frac{\sin\alpha\cos(\alpha-\beta)+\cos\alpha\sin(\alpha-\beta)}{\cos\alpha\sin(\alpha-\beta)}\right).$$

$$R_1R_2=\frac{v^4}{g^2}\sin 2\alpha\sin 2(\alpha-\beta)=\frac{4v^4}{g^2}\sin\alpha\cos\alpha\sin(\alpha-\beta)\cos(\alpha-\beta)$$

$$=\frac{a^4}{c^2}\frac{\sin\alpha\cos(\alpha-\beta)}{\cos\alpha\sin(\alpha-\beta)}.$$

$$\therefore a(R_1+R_2)-R_1R_2=\frac{a^4}{c^2}\frac{\cos\alpha\sin(\alpha-\beta)}{\cos\alpha\sin(\alpha-\beta)}=\frac{a^4}{c^2}.$$

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

Edited by Dr. G. E. Wahlin, University of Illinois.

183. Proposed by MERTON T. GOODRICH, Dixfield, Maine.

What relations must exist between the quantities A, B, and C in the harmonic ratio $\frac{AB}{(A+B+C)(-C)}=-1$ so that they will be positive integers.

II. Solution by the PROPOSER.

Solving the given equation for A, we have $A=\frac{(B+C)C}{B-C}$. Since A, B, and C are positive, we see that $B-C$ must be positive. Since A is an integer, either $\frac{C}{B-C}=\frac{M}{N}=K$, or $\frac{B+C}{B-C}=\frac{M'}{N'}=K'$, where M and N, M' and N' are integers, and M prime to N, and M' prime to N'. From the first of these equations, $C=K(B-C)$. Since C and $B-C$ are positive, K must be positive. Solving this last equation for B, we have $B=\frac{(K+1)C}{K}=\frac{(M+N)C}{M}$. Since M and N are relatively prime, $M+N$ is prime to M. Hence, B being an integer, C is divisible by M. That is, $C=MD'=MND'/N=KD$, where $D=ND'$, and D' and hence D are positive integers.

Substituting KD for C in the expression for B, $B=(K+1)D=C+D$. Substituting KD for C and $(K+1)D$ for B in the expression for A, $A=(2K+1)KD=(2K+1)C$. Putting $\frac{M}{N}$ for K, $A=\frac{(2M+N)MD}{N^2}$. This tells us that if N is odd $D=N^2.\lambda$; but if N is even, $2M+N$ is even and then $D=\frac{N^2.\lambda}{2}$. Hence we have this set of relations: $A=(2K+1)C$, $B=C+D$,